INDECOMPOSABLE HIGHER CHOW CYCLES ON LOW DIMENSIONAL JACOBIANS

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There is a basic indecomposable higher cycle $K \in CH^g(J(C),1)$ on the Jacobian $J(C)$ of a general hyperelliptic curve C of genus g, see [3]. Consider K_t the translation of K associated with a point $t \in C$, we prove that in general $K-K_t$ is indecomposable if $g \geq 3$. Our tool is Lewis' condition for indecomposability [8]. We also show that on the jacobian $J(C)$ of any curve C of genus 3 there is a geometrically natural family of higher cycles, when C becomes hyperelliptic the family in the limit contains a component of indecomposable cycles of type $K - K_t$.

Introduction

The first higher Chow group $CH^p(X,1) \simeq H^{p-1}(X, \mathcal{K}_p)$ of a non singular variety X is generated by higher cycles of the form $Z = \sum_i Z_i \otimes f_i$, where the Z_i are irreducible subvarieties of codimension $(p-1)$ and the rational functions $f_i \in \mathbb{C}(Z_i)^\times$ obey the rule $\sum_i div(f_i) = 0$ as a cycle on X. Consider the subgroup of decomposable cycles

$$CH^k_{\rm dec}(X,1) := Im\big\{CH^1(X,1)\otimes CH^{k-1}(X) \longrightarrow CH^k(X,1)\big\} \quad ,$$

and the related quotient of indecomposable cycles

$$CH^k_{\rm ind}(X,1;\mathbb{Q}) := \big(CH^k(X,1)/CH^k_{\rm dec}(X,1)\big)\otimes\mathbb{Q} \quad .$$

A geometrically natural higher cycle K for $CH^g(J(C),1)$ was given in [3], here $J(C)$ is a hyperelliptic Jacobian, and it was proved that K is indecomposable when C is general. This result was reached by showing that the interesting part of the regulator image of K must be non trivial. The regulator or cycle class is a map $cl_{k,1}$ from the higher Chow groups into Deligne cohomology [1],

$$cl_{k,1} : CH^k(X,1;\mathbb{Q}) \longrightarrow H^{2k-1}_{\mathcal{D}}(X, \mathbb{Q}(k))\,,$$

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which satisfies a rigidity property of Beilinson type [10], namely $CH^k_{\text{ind}}(X,1;\mathbb{Q})$ has countable image modulo 'Hodge classes'. Our aim here is to show that translation on J acts non trivially on K when the genus of C is (at least) 3, to the effect that the orbit of K in $CH^g_{\text{ind}}(J(C),1;\mathbb{Q})$ is uncountable. We use in an essential way Lewis' condition for indecomposability [8]. Voisin has conjectured some time ago that for a smooth projective variety X the group $CH^2_{\text{ind}}(X,1;\mathbb{Q})$ is countable [11], we verify below that indeed in genus 2 translation does not change the indecomposable class of K.

Lewis' conditions are some Hodge-theoretic properties of the regulator image of a complete family of higher cycles on X, when they hold the general member in the family is indecomposable, even if its regulator class vanishes. The main result of [6] shows that indeed there is a complete family satisfying the conditions, in their example X is a sufficiently general product of three elliptic curves. The cycles involved may appear to be ad-hoc, this has moved me to consider certain geometrically natural cycles, the '4-configurations', which live on the general jacobian of genus 3. By specializing to the hyperelliptic locus a 4-configuration becomes the hyperelliptic configuration $K-K_t$, that is the difference of the basic cycle by its translation associated with a point $t\in C$. This leads us to conjecture that in general the 4-configuration is indecomposable.

In the last part we show that translation on a bielliptic jacobian $J(G)$ acts non trivially on the indecomposable type of some cycles in $CH^3(J(G),2)$. This is in contrast with the known fact that on an elliptic curve E translation acts trivially on $CH^2(E,2)\simeq H^0(E,\mathcal{K}_2)$ modulo $K_2(\mathbb{C})$.

1. Lewis' condition.

In this section we recall some notation and definitions and then we state one of Lewis' theorems. Our aim is to have a concrete reference at hand, for more details the reader should consult either the original paper [8] or the excellent survey [7].

1.1 The real regulator and some definitions.

Let X be projective and nonsingular, it is

$$\frac{H^{i-1}(X,\mathbb{C})}{F^jH^{i-1}(X,\mathbb{C})+H^{i-1}(X,\mathbb{R}(j))}\simeq\frac{H^{i-1}(X,\mathbb{R}(j-1))}{\pi_{j-1}(F^jH^{i-1}(X,\mathbb{C}))},$$

and therefore one has

$$\begin{aligned}H^{2k-1}_{\mathcal{D}}(X,\mathbb{R}(k))&\simeq H^{2k-2}(X,\mathbb{R}(k-1))\cap F^{k-1}H^{2k-2}(X,\mathbb{C})\\&=: H^{k-1,k-1}(X,\mathbb{R}(k-1)).\end{aligned}$$

According to Beilinson [1] the real regulator image of a cycle $Z\in CH^k(X,1;\mathbb{Q})$ is the element

$$R_{k,1}(Z)\in H^{2k-1}_{\mathcal{D}}(X,\mathbb{R}(k))\simeq H^{k-1,k-1}(X,\mathbb{R}(k-1)),$$

determined by the class of the current

$$R_{k,1}(Z):\omega\longmapsto(2\pi\sqrt{-1})^{k-1-d}\sum_i\int_{Z_i-Z_i^{\text{sing}}}\omega\,\log|f_i|\ .$$

Definition: level of a quotient $CH^k(X,m;\mathbb{Q})/G$.
Consider a subgroup G of $CH^k(X,m;\mathbb{Q})$, the *level* of $CH^k(X,m;\mathbb{Q})/G$ is the least nonnegative integer r such that

$$CH^k(X,m;\mathbb{Q}) = G + i_*CH^{r+m}(Y,m;\mathbb{Q})$$

for some $i : Y \hookrightarrow X$ closed and of pure codimension $(k-r-m)$.

Definition: $H_N^{k-l,k-m}(X)$.
The *coniveau filtration* is given by

$$N^jH^i(X,\mathbb{Q}) := \ker\big(\, H^i(X,\mathbb{Q}) \longrightarrow lim_{codim_X Y \geq j} H^i(X-Y,\mathbb{Q})\,\big),$$

where the direct limit is over closed subvarieties $Y \subset X$. We let

$$H_N^{k-l,k-m}(X) := Im(\, N^{k-l}H^{2k-l-m}(X,\mathbb{Q}) \otimes \mathbb{C} \longrightarrow H^{k-l,k-m}(X)\,) \quad ,$$

and we note that it is the complex subspace generated by the Hodge projected image of the coniveau filtration.

Fact. Lewis constructs certain complex subspaces

$$H^{\{k,l,m\}}(X) \subseteq H^{k-l,k-m}(X) \quad ,$$

such that for $m = 0$ one has

$$H^{\{k,l,0\}}(X) \subseteq H_N^{k-l,k}(X)$$

The spaces $H^{\{k,l,m\}}(X)$ are obtained by a process of Künneth projection of the Hodge components of the real regulator classes of the elements in $CH^k(X \times S; m)$, with varying S, S projective and nonsingular.

1.2 Lewis's theorem. We recall Lewis's theorem in an abridged version, because we need it only for $CH^k_{\mathrm{ind}}(X,1;\mathbb{Q})$.

Theorem ([8]). *Let X be a projective algebraic manifold. Then*
(1) $H^{\{k-1,l-1,0\}}(X) \subset H^{\{k,l,1\}}(X)$.
(2) *If* $H^{\{k,l,1\}}(X)/H^{\{k-1,l-1,0\}}(X) \neq 0$ *then* $level(CH^k_{\mathrm{ind}}(X,1;\mathbb{Q})) \geq l-1$.

Corollary. *If $H^{\{k,l,1\}}(X)/H^{\{k-1,l-1,0\}}(X) \neq 0$ for some l with $2 \leq l \leq k$, then $CH^k_{\mathrm{ind}}(X,1;\mathbb{Q})$ is uncountable.*

We shall prove that the hypotheses of the corollary hold when X is the general hyperelliptic jacobian of genus 3 for $k=3$, $l=2$.

2. The basic hyperelliptic construction.

2.1. The hyperelliptic configuration. Let $f : C \to \mathbb{P}^1$ be the double cover associated with a hyperelliptic curve, we fix two ramification points w_1 and w_2 on C and choose a standard parameterization on $\mathbb{P}^1$ so that $f(w_1) = 0$ and $f(w_2) = \infty$. The points w_1 and w_2 are for us the distinguished Weierstrass points, and $\epsilon := class(w_1 - w_2)$ is the associated element of order two in $Pic^0(C)$.

It is convenient for us to identify $J(C) = Pic^0(C)$ with $Pic^1(C)$ by adding w_1. We embed C in the natural way in $Pic^1(C)$, for $t \in Pic^1(C)$ we let C_t be the translate of C by class$(t - w_1)$. Consider now $W_1 := C = C_{w_1}$ and $W_2 := C_{w_2}$, the ϵ translate of C, and fix a point $t \in C$. It is useful to observe that the intersection $C_t \cap W_1 \cap W_2$ is the point w_1. We shall follow the convention to indicate a rational function on C_t by using the same name given to the corresponding function on C.

Consider $K := W_1 \otimes f + W_2 \otimes f$ and its t-translation $K_t := C_t \otimes f + C_{t+\epsilon} \otimes f$. Now K is the basic cycle of [3], hence it is a non trivial indecomposable element of $CH^g(J(C), 1; \mathbb{Q})$ for C general enough. Here we are interested in the cycle $Z_t := K - K_t$, which we call the hyperelliptic configuration. It is clear that Z_t belongs to the kernel of the regulator map, because the regulator at hand is computed as a current acting on 2-forms on J, hence it is indifferent to translation. Our aim is to prove that for general t the hyperelliptic configuration Z_t is indecomposable if the genus is at least 3, but first we show that on the contrary

Proposition. *In genus* 2 *the hyperelliptic configuration is decomposable.*

The divisor class of $W_1 - W_2$ is of order 2 in $Pic^0(J(C))$ hence there is a rational function θ, say, with $div(\theta) = 2W_1 - 2W_2$, and we may assume without restriction that θ is symmetric with respect to the ϵ involution on $J(C)$, namely that $\epsilon^*\theta = \theta^{-1}$. The other two curves in the support of the configuration C_t and $C_{t+\epsilon}$ are the translation respectively of W_1 and W_2 and thus $2(C_t - C_{t+\epsilon}) = div(\theta_t)$, where θ_t is also symmetric. The divisor of $(\theta_t)_{|W_1}$ is supported on W_1 at the distinguished Weierstrass points because at the other point t of intersection of W_1 with C_t the contributions cancel out, and in fact $(\theta_t)_{|W_1} = c_t f$ for some nonzero constant c_t, depending on t, because we see that $(\theta_t)_{|W_1}$ and f have the same divisor.

By symmetry $(\theta_t)_{|W_2} = (c_t f)^{-1}$, and also $(\theta)_{|C_t} = k_t f$ and $(\theta)_{|C_{t+\epsilon}} = (k_t f)^{-1}$. By definition the tame symbol of two rational functions a and b yields at a divisor D the rational function $(T\{a, b\})_D = (-1)^{\nu_D(a)\nu_D(b)}(a^{\nu_D(b)}/b^{\nu_D(a)})_{|D}$ where ν_D is the order at D. In this way it is

$$T\{\theta_t, \theta\} = 2(W_1 \otimes f + W_2 \otimes f + C_t \otimes f^{-1} + C_{t+\epsilon} \otimes f^{-1} + (W_1 + W_2) \otimes c_t - (C_t + C_{t+\epsilon}) \otimes k_t),$$

and therefore Z_t vanishes in $CH^2_{\text{ind}}(J(C), 1; \mathbb{Q})$.

2.2. A higher Chow cycle on $J \times C$. We keep the identification $J(C) \simeq Pic^1(C)$. We embed $C \times C$ in $J \times C$ in four different ways, two of them twisted and two of them straight. The straight surfaces are $S_1 := W_1 \times C$ and $S_2 := W_2 \times C$. The twisted ones are T_1 and T_2, they are the images in $J \times C$ of the maps which send a point (x, t) in $C \times C$ to $(class(x + t - w_1), t)$ and $(class(x + t - w_2), t)$ respectively. Consider the rational function F on $C \times C$ which is the pull-back under the first projection of f on C. We have on S_i and T_i rational functions F_i and G_i respectively, they correspond to F under the said isomorphisms.

In this manner $\mathfrak{S} := S_1 \otimes F_1 + S_2 \otimes F_2$ and $\mathfrak{T} := T_1 \otimes G_1 + T_2 \otimes G_2$ are higher Chow cycles for $CH^g(J \times C, 1; \mathbb{Q})$. Our final cycle is

$$\mathfrak{Z} := \mathfrak{S} - \mathfrak{T} \qquad .$$

Taking sections over $t \in C$ yields $\mathfrak{S}_*(t) = K$ and $\mathfrak{T}_*(t) = K_t$, and thus we have $\mathfrak{Z}_*(t) = Z_t$, the hyperelliptic configuration.

2.3. Lewis' method and a theorem of Lieberman.

Lewis' conditions deal with the Hodge-Künneth decomposition of the real regulator image of a higher cycle like $\mathfrak{Z}$, if they are satisfied then the general section $\mathfrak{Z}_*(t)$ is indecomposable. We prove below that this is the case in our situation and thus we have that but possibly for a countable subset of points in C:

2.3.1. Theorem. *The hyperelliptic configuration Z_t is indecomposable*

In order to proceed we need to recall that the General Hodge Conjecture asserts that the filtration on cohomology with rational coefficients induced by the Hodge filtration should coincide with the coniveau filtration. Lieberman [9] proved that the General Hodge Conjecture holds for $H^3(J)$, when J is a general hyperelliptic jacobian of genus 3. More precisely the only subhodge structures in $H^3(J)$ are the primitive part P^3 and the product $\Theta H^1(J)$, Θ being the class of the theta divisor. In this way the coniveau 1 filtration $N^1H^3(J)$ is exactly $\Theta H^1(J)$ and therefore it is orthogonal to P^3. This is the point where we need to use the hypothesis $g = 3$. Pirola (oral communication) claims that the analogue of Lieberman's result holds in fact for any general hyperelliptic jacobian of genus at least 3. If this is the case then our proof would yield indecomposability of Z_t also for $g \geq 4$.

In our situation Lewis' condition is fulfilled if we prove that there exists a certain class $\eta \in H^{2,1}(J)$ and a form $\nu \in H^{0,1}(C)$ such that the pairing $< R(\mathfrak{Z}), \eta \wedge \nu > \neq 0$, where $R(\mathfrak{Z})$ is the real regulator image of $\mathfrak{Z}$ defined as a current. In order to satisfy Lewis' requirement η must be orthogonal to $H^{\{2,1,0\}}(J)$, and this is the case when η is primitive, because of Lieberman's result.

2.4. On the real regulator image of K.

The standard inner product on the space $H^0(C, K_C)$ of global holomorphic forms on a curve C is $< \alpha, \beta > := (i/2) \int_C \alpha \wedge \bar{\beta}$. We fix a basis $\omega_i^J, \ldots, \omega_g^J$ for the 1-forms on J with the property that $\zeta_i := a^* \omega_i^J$ is an orthonormal basis for $H^0(C, K_C)$, here $a : C \to J$ is the standard map. The class of the theta divisor on J is then given by the form $\theta_J = (i/2) \sum_{j=1}^g \omega_j^J \wedge \bar{\omega}_j^J$.

When C is hyperelliptic and otherwise general then the basic cycle K is indecomposable because the primitive contribution of the standard regulator image of K does not vanish, see [3]. This information is found by the study of an infinitesimal invariant of Griffiths type. For our purpose here we need a more concrete computation on the behaviour of the *real* regulator image of K. Let $\tau := \omega_1^J \wedge \bar{\omega}_1^J - \omega_2^J \wedge \bar{\omega}_2^J$, then

2.4.1. Proposition. $< R(K), \tau > \neq 0$.

The proof is given below in (2.6).

2.5. Lewis' condition holds for the regulator image of $\mathfrak{Z}$. We take $g = 3$ and C general enough so that Lieberman's result applies to $J(C)$. It is straightforward to check that $\eta := \tau \wedge \omega_3^J$ gives a primitive class in $H^{2,1}(J(C))$ and thus η is orthogonal to the coniveau 1 filtration. Let ν be the antiholomorphic form on $J \times C$ which is the pull-back of $\bar{\zeta}_3 = a^* \bar{\omega}_3^J$ from the second factor C. We have

2.5.1. Theorem. $< R(\mathfrak{Z}), \eta \wedge \nu > \neq 0$.

Proof. It is $< R(\mathfrak{Z}), \eta \wedge \nu > = < R(\mathfrak{S}), \eta \wedge \nu >$ - $< R(\mathfrak{T}), \eta \wedge \nu >$. To compute we pull back $\eta \wedge \nu$ to $C \times C$ by means of the 4 relevant maps $C \times C \to J \times C$. In the

straight case, that is $\mathfrak{S}$, the pull back of η is the zero form, because a 3-form dies on the curve W_i, and therefore $< R(\mathfrak{S}), \eta \wedge \nu >= 0$.

The twisted embedding with image T_1 is the map $g : C \times C \to J \times C$ defined as $g(x,t) = (g_1(x,t), t)$, where $g_1 : C \times C \to J$ is the function $g_1(x,t) = class(x + t - w_1)$. It is $g_1^*(\omega_n^J) = \zeta_n^1 + \zeta_n^2$, here the superscript i indicates pull-back under the i-th projection $C \times C \to C$. Keeping this notation we obtain:

$$g^*(\eta \wedge \nu) = (\zeta_1^1 \wedge \bar{\zeta}_1^1 - \zeta_2^1 \wedge \bar{\zeta}_2^1) \wedge (\zeta_3^2 \wedge \bar{\zeta}_3^2) + \sum_i \alpha_i^1 \wedge \beta_i^2$$

where the forms β_i^2 are pull-back from the second factor of forms β_i with the property $\int_C \beta_i = 0$. Our aim is to check the non vanishing of

$$\int_{T_1} \eta \wedge \nu \, \log |G_1| \; + \int_{T_2} \eta \wedge \nu \, \log |G_2|$$

going to $C \times C$ this is twice

$$\int_{C \times C} g^*(\eta \wedge \nu) \, \log |F| \; ,$$

Now F is the pull-back of the rational function f from the first factor C and therefore we can use the product structure to calculate the integrals. One has

$$\int_{C \times C} g^*(\eta \wedge \bar{\zeta}_3) \, \log |F| =$$
$$(\int_C (\zeta_1^1 \wedge \bar{\zeta}_1^1 - \zeta_2^1 \wedge \bar{\zeta}_2^1) \log |f|)(\int_C \zeta_3^2 \wedge \bar{\zeta}_3^2) + \sum_i (\int_C \alpha_i^1 \log |f|)(\int_C \beta_i^2) \; =$$
$$(\int_C (\zeta_1^1 \wedge \bar{\zeta}_1^1 - \zeta_2^1 \wedge \bar{\zeta}_2^1) \log |f|)(\int_C \zeta_3^2 \wedge \bar{\zeta}_3^2) \qquad .$$

Up to a nonzero constant this is precisely $< R(K), \tau >$, the regulator pairing for the basic cycle, hence it is not zero by proposition 2.4.1.

2.6. Useful integrals and the proof of 2.4.1.

The proof of 2.4.1 depends to a large extent on a reduction process to the case of elliptic curves, as we explain next.

Let E_λ be the elliptic curve with affine equation $y^2 = x(x-1)(x-\lambda)$. Then we define $f_\lambda := x$ as a rational function on E_λ. Let $\theta_\lambda = (i/2)\omega_\lambda \wedge \bar{\omega}_\lambda$ be the invariant volume form on E_λ. We define $I(\lambda) := \int_{E_\lambda} \log |f_\lambda| \theta_\lambda$.

2.6.1. Proposition. *$I(\lambda)$ varies with λ.*

Proof. Multiplication of x by λ^{-1} shows that λ and λ^{-1} determine the same curve E, say. On E the volume forms coincide, while $f_\lambda = \lambda f_{\lambda^{-1}}$. In this way

$$I(\lambda) = \int_E \log |\lambda| \theta + I(\lambda^{-1}) = \log |\lambda| + I(\lambda^{-1})$$

hence $I(\lambda)$ cannot be constant.

We move to the case of genus 2 and define τ as in 2.4. Consider the hyperelliptic map $f : C \to \mathbb{P}^1$ and define $I(f, \tau) := \int_C \log |f| \tau$.

2.6.2. Proposition. *If C is general and of genus 2 then $I(f,\tau) \neq 0$.*

Proof. We prove this for the genus two bielliptic curve C which is a double cover of $E(1) := E_{\lambda_1}$ and of $E(2) := E_{\lambda_2}$. Consider the diagram

$$\begin{tikzcd} E_2 \arrow[d, "f_2"'] & C \arrow[l, "k_2"'] \arrow[r, "k_1"] \arrow[d, "f"'] & E_1 \arrow[d, "f_1"'] \\ \mathbb{P}^1 & \mathbb{P}^1 \arrow[l, "h"'] \arrow[r, "h"] & \mathbb{P}^1 \end{tikzcd}$$

Here f_i is ramified over $(0, 1, \infty, \lambda_i)$, h is the double cover ramified over λ_1 and λ_2, and $f : C \to \mathbb{P}^1$ is the hyperelliptic cover ramified at $h^{-1}(\{0, 1, \infty\})$. On the range of h we have already fixed a standard parameter, we choose a standard parameter on the domain of h so that 0 maps to 0, and similarly for 1 and for ∞. In this manner f is a well defined rational function on C, and we denote by $\bar{f}$ its transform under the involution of $\mathbb{P}^1$ associated with h. Consider the composition $g := hf$, then g is a rational function of degree 4 on C and in fact $f\bar{f} = cg$, c a non-zero constant.

Given a rational function r on C we set for $i = 1$ or $= 2$

$$I(r, i, C) := \int_C k_i^*(\theta_i) \log |r| \,, \qquad\qquad I(r, \theta_1 - \theta_2) := I(r, 1, C) - I(r, 2, C)$$

Using the previous notations we see that $I(g, i, C) = I(\lambda_i)$, and therefore in general it is $I(g, \theta_1 - \theta_2) \neq 0$. We have

$$I(f, \theta_1 - \theta_2) + I(\bar{f}, \theta_1 - \theta_2) = I(g, \theta_1 - \theta_2) + log|c| \int_C (k_1^*(\theta_1) - k_2^*(\theta_2)) \; = I(g, \theta_1 - \theta_2)$$

and therefore $I(f, \theta_1 - \theta_2) \neq 0$ for the general bielliptic curve C. Recall now that $(1/2)(k_1^*(\theta_1) - k_2^*(\theta_2))$ can be taken to be τ_C and then conclude that $I(f, \tau) \neq 0$.

To complete the proof of 2.4.1 we show now that it holds for the case of a curve C which is an unramified double cover $\pi : C \to G$, where G is a general curve of genus 2. To begin we remark that $\pi^* \tau_G$ is a multiple of a form on C which can be taken as τ_C. There is a diagram

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[d, "f"'] & G \arrow[d, "f_G"'] & C \arrow[l, "\pi"'] \arrow[d, "k_E"'] \\ \mathbb{P}^1 \arrow[r, "h"] & \mathbb{P}^1 & E \arrow[l, "f_E"'] \end{tikzcd}$$

here $\pi : C \to G$ is étale, $f_G : G \to \mathbb{P}^1$ is the hyperelliptic cover, ramified at 6 points, $\{0, 1, \infty, \lambda, a_1, a_2\}$. The cover h is ramified at a_1, a_2 and $f_E : E \to \mathbb{P}^1$ is ramified at the other points. The map $f : C \to \mathbb{P}^1$ is also an hyperelliptic cover, it is ramified over the pull-back via h of the ramification points of f_E, and finally $k_E : C \to E$ is ramified along the pull back via f_E of the ramification of h. We set here $g := hf$ and define the parameter on the domain of h in such a way that $h(0) = 0$, $h(1) = 1$, $h(\infty) = \infty$. Using 2.6.2. for G we conclude by a similar argument that 2.4.1 holds.

3. Natural higher cycles on the general jacobian of genus 3

3.1. The 4-configuration.

By 4-configuration we mean here a certain cycle in $CH^3(J(C),1)$ which is supported on 4 copies of the curve C. The configuration depends on the choice of points a' and a'' and p' and p'' on C with the condition that $class((a'+a'')-(p'+p''))=\epsilon$ is a torsion 2 element in $J(C)$. More precisely we choose a rational function f with $div(f)=2((a'+a'')-(p'+p''))$. We embed C in Pic^3 in 4 ways as follows. Let $i(y,z):C\to Pic^3$ be the map $i(y,z)(x)=x+y+z$, and $C(y,z):=i(y,z)(C)$, and let $j:C\to Pic^3$ be the map $j(x)=-x+2(a'+a'')$, $G:=j(C)$. We consider $C(a',a'')$, $C(p',p'')$ and G. Translation by ϵ maps $C(a',a'')$ to $C(p',p'')$, and we take G_ϵ to be the image of G. We shall use the convention that f represents the rational function on each of the preceding curves which maps to f under the chosen isomorphism with C and thus we set $Z_1:=C(a',a'')\otimes f$, $Z_2:=G\otimes f$, $Z_3:=C(p',p'')\otimes f$, $Z_4:=G_\epsilon\otimes f$.

Proposition. $Z:=\sum_{i=1}^4(-1)^iZ_i$ *is a higher cycle.*

The only possible difficulty is to see where the curves intersect. Now $C(a',a'')$ intersects G in two points, on both curves the points come from a' and a'' under the isomorphism with C, but the point which comes from a' in G comes from a'' in $C(a',a'')$ and conversely. A similar statement holds for $C(p',p'')\cap G$, and then intersections with G_ϵ can be recovered by using ϵ-symmetry. Note that if C is not hyperelliptic then $C(a',a'')\cap C(p',p'')=\emptyset$ and $G\cap G_\epsilon=\emptyset$.

3.2. The hyperelliptic configuration is a special kind of 4-configuration.

It is apparent that there is a 1-dimensional family of 4-configurations associated with a given 2 torsion class ϵ for a fixed curve C of genus 3. We specialize the curve to be hyperelliptic. We degenerate also the given pencil g^1_4, by the request that a' and p', say, move to become the same point t on the hyperelliptic curve C, and therefore a'' becomes the Weierstrass point w_1 and p'' is w_2. In this manner the rational function f must be the Weierstrass one and the pencil g^1_4 is 'degenerate', namely g^1_2 $+2t$. Strictly speaking our construction should be called deformation *from* the hyperelliptic curve. In this hyperelliptic case we choose to denote by $Z(t)$ the higher cycle which is constructed according to the rule of the 4-configuration so to make explicit its dependence on t. It is simple to check that $Z(t)$ is indeed the hyperelliptic configuration $K-K_t$ of section 1.2, and therefore it is indecomposable. This is best seen by identifying $Pic^3\to Pic^1(C)=J(C)$ by first mapping $Pic^3\to Pic^{-3}$ and then going from Pic^{-3} to Pic^1 by adding the divisor $2(t+w_1)$. In this manner G and G_ϵ become W_1 and W_2 respectively, while the curve $C(t,w_1)$ goes to C_t and $C(t,w_2)$ to $C_{t+\epsilon}$. There is apparently a minor problem here, in that the chosen isomorphism of C with C_t differs from the isomorphism $C\to C(t,w_1)\to C_t$, by the hyperelliptic involution. This fact is of no consequence, because the Weierstrass function is of course invariant under the involution.

3.2.1. Remark. We have seen that on the general Jacobian the 4-configuration yields 4 curves and 8 points of intersection. Each curve does not intersect its ϵ-translate, and meets the remaining curves each in 2 points. Each point belongs to 2 curves. On the other hand when the configuration becomes the hyperelliptic one just described, then the points of intersection become 4 in all, every curve meets every other in 2 points, but now each point belongs to 3 curves and each curve to 3 points.

I do not know how to turn into a proof the heuristic observation that the indecomposability of the hyperelliptic configuration should yield indecomposability for the general 4-configuration, still I think that there is enough evidence for:

3.2.2 Conjecture. *The general 4-configuration is indecomposable.*

A strong motivation in this direction is Fakhruddin's work [4], indeed our theorem (2.3.1) and the conjecture may both be interpreted as a kind of extension to higher Chow groups of some of Fakhruddin's results.

4. Translations act non trivially on $CH^{g+1}_{\mathrm{ind}}(J(C),2)$.

We consider now the second higher Chow group $CH^p(X,2) \simeq H^{p-2}(X,\mathcal{K}_p)$, according to the terminology of [8] the group of decomposable cycles is here

$$CH^k_{\mathrm{dec}}(X,2) := Im\big\{K_2(\mathbb{C})\otimes CH^{k-2}(X) \longrightarrow CH^k(X,2)\big\} \quad ,$$

while the indecomposable group is

$$CH^k_{\mathrm{ind}}(X,2;\mathbb{Q}) := \big(CH^k(X,2)/CH^k_{\mathrm{dec}}(X,2)\big)\otimes\mathbb{Q} \quad .$$

It is known that translations on an elliptic curve E act trivially on $CH^2_{\mathrm{ind}}(E,2)$, see [5, 3.10]. We show that on the contrary translations on a genus 2 bielliptic jacobian $J(G)$ operate non trivially on $CH^3_{\mathrm{ind}}(J(G),2)$. Our procedure is similar to the one used above for the first higher Chow group. The hyperelliptic configuration is replaced by a certain B-configuration, which is then shown to be indecomposable by checking Lewis' condition on a cycle $\mathfrak{B}$ of $CH^3(J(G)\times G,2)$.

4.1. The bielliptic configuration.
Spencer Bloch defined and studied for the first time higher regulator maps. In his seminal memory [2] he constructed certain elements $S_b \in \Gamma(E,\mathcal{K}_2))$ associated with a point b of finite order on an elliptic curve E. It is one consequence of his deep work that the real regulator image of S_b is not trivial for some curves with complex multiplication, and therefore that it is not trivial in general. In [3] it is shown that moreover $CH^2_{\mathrm{ind}}(E,2)$ is not finitely generated.

Consider a bielliptic curve G of genus 2 with associated map $\delta_G : G \to E_1$, and let $a : G \to J(G)$ be the Abel Jacobi map. In this way $Z(b) := a_*\delta_G^*(S_b)$ is a cycle in $CH^3(J(G),2)$. Translation by an element $t \in Pic^0(G)$ maps $Z(b)$ to the cycle $Z_t(b)$, our aim is to prove

4.1.1. Theorem. *The bielliptic configuration $B(t) := Z_t(b) - Z(b)$ is indecomposable in general.*

Note that $B(t)$ has trivial regulator image.

Proof. Consider the cycle $G\times Z(b)$ as an element in the higher Chow group of $G\times G$. The straight embedding $\sigma := id\times a : G\times G \to G\times J(G)$ maps it to $\mathfrak{S} := \sigma_*(G\times Z(b))$ in $CH^3(G\times J(G),2)$. The twisted embedding $\tau(t,x) := (t, a(x) + (t-w_1)))$ gives instead $\mathfrak{T} := \tau_*(G\times Z(b))$, with section $\mathfrak{T}_*(t) = Z_t(b)$, and therefore $B(t)$ is the section at $t \in G$ of $\mathfrak{B} := \mathfrak{T} - \mathfrak{S}$.

We use the same type of notations as we did in part 2, in particular the holomorphic form ω_i^J comes from E_i, we need to consider also the forms $\nu := \bar{\omega}_1^J \wedge \omega_2^J$ on $J(G)$ and $\bar{\zeta}_2$ on G. The procedure of 2.5.1 gives here again $< R(\mathfrak{B}), \bar{\zeta}_2 \wedge \nu > \neq 0$.

The Neron Severi space of divisors with rational coefficients on $J(G)$ is isomorphic to the same space on the product of the two associated elliptic curves. On the general bielliptic jacobian ν is orthogonal to the Neron Severi group, because it is orthogonal to the elliptic curves. For this reason the Künneth projected image of $R(\mathfrak{B})$ yields a non trivial element in $H^{\{3,2,2\}}(J(G))/H^{\{1,0,0\}}(J(G))$. A look at the proof of the main theorem of [8] shows that this fact implies that the general section $\mathfrak{B}_*(t)$ is indeed indecomposable.

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